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6 SEM TDC MTMH (CBCS) C 14

2023

(May/June)

MATHEMATICS

(Core)

Paper : C-14

(Ring Theory and Linear Algebra—II)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer any *three* from the following : 5×3=15

   (a) Prove that a ring R is a commutative ring with unity if and only if the corresponding polynomial ring $R[x]$ is commutative with unity.

   (b) If F is a field, then prove that the polynomial ring $F[x]$ is not a field.

(c) Write about irreducibility of a polynomial. Test the irreducibility of the following polynomials : 1+2+2=5

(i) $f(x) = 3x^5 + 15x^4 - 20x^3 + 10x + 20$, over Q

(ii) $f(x) = 21x^3 - 3x^2 + 2x + 9$, over Z_2

(d) Define principal ideal domain and prove that in a principal ideal domain, an element is an irreducible iff it is prime. 1+4=5

2. Answer any *three* of the following : 5×3=15

(a) Define unique factorization domain (UFD) and prove that every field is unique factorization domain. 1+4=5

(b) Prove that the ring of Gaussian integer $Z[i] = \{a + ib | a, b \in Z\}$ is Euclidian domain.

(c) Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in Z[x]$. If there is a prime such that $p \nmid a_n$, $p | a_{n-1}, \cdots, p | a_0$ and $p^2 \nmid a_0$, then prove that $f(x)$ is irreducible over Q.

(d) Prove that every Euclidian domain is a principal ideal domain.

3. Answer any *three* of the following : 6×3=18

(a) Let V be a finite dimensional vector space over the field F. If α is any vector in V, the function L_α of V^* defined by $L_\alpha(f) = f(\alpha)$, $\forall f \in V^*$, then prove that L_α is a linear functional and the mapping $\alpha \to L_\alpha$ is an isomorphism of V onto V^{**}.

(b) Determine the eigenvalues and the corresponding eigenspaces for the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

(c) Show that similar matrices have the same minimal polynomial. Also, find the minimal polynomial for the real matrix

$$\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$$

(d) Let V be a finite dimensional vector space over the field F and W be a subspace of V. Then prove that

$$\dim W + \dim W^{\circ} = \dim V$$

4. (a) Let $T : R^2 \to R^2$ be a linear operator defined by

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Then find all the T-invariant subspace of $R^2(R)$. 2

(b) Let T be a linear operator on R^3 which is represented in the standard basis by the matrix

$$\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$$

Prove that T is diagonalizable. 5

Or

If T is a linear operator on a vector space V and W is any subspace of V, then prove that $T(W)$ is a subspace of V. Also show that W is invariant under T iff $T(W) \subseteq W$.

5. (a) If V is inner product space, then for any vectors $\alpha, \beta \in V$ and any scalar c, prove that—

(i) $||\alpha|| > 0$ for $\alpha \neq 0$

(ii) $||c\alpha|| = |c|\,||\alpha||$

(iii) $|(\alpha|\beta)| \leq ||\alpha||\,||\beta||$ 5

(b) Apply Gram-Schmidt process to the vectors $\beta_1 = (1, 0, 1)$, $\beta_2 = (1, 0, -1)$, $\beta_3 = (0, 3, 4)$ to obtain an orthonormal basis for $V_3(R)$ with the standard inner product. 5

(c) Let W be any subspace of a finite dimensional inner product space V and let E be the orthogonal projection of V on W. Prove that $V = W + W^{\perp}$, where $W^{\perp}$ is the null space of E. 5

Or

If $B = \{\alpha_1, \alpha_2, \cdots, \alpha_m\}$ is any finite orthonormal set in an inner product space V and if β is any vector in V, then prove that

$$\sum_{i=1}^{m} |(\beta, \alpha_i)|^2 \leq ||\beta||^2$$

6. (a) Define orthogonal set. If α and β are orthogonal unit vectors, then write the distance between them. 1+1=2

(b) Answer any *two* of the following : 4×2=8

(i) Let T be a linear operator on R^2, defined by $T(x, y) = (x + 2y, x - y)$. Find the adjoint T^*, if the inner product is standard one.

(ii) Let V be a finite dimensional inner product space and let $B = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ be an ordered orthonormal basis for V. Let T be a linear operator on V. Let $A = [a_{ij}]_{n \times n}$ be the matrix of T with respect to ordered basis B, then prove that $a_{ij} = (T\alpha_j, \alpha_i)$.

(iii) Let $S = \{\alpha_1, \alpha_2, \cdots, \alpha_m\}$ be an orthogonal set of non-zero vectors in an inner product space V. If a vector β in V is in the linear span of S, then show that

$$\beta = \sum_{k=1}^{m} \frac{(\beta, \alpha_k)}{||\alpha_k||^2} \alpha_k$$

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