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6 SEM TDC MTMH (CBCS) C 14

2025

(May)

MATHEMATICS

(Core)

Paper : C–14

(Ring Theory and Linear Algebra—II)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. *(a)* If F is commutative, then write the condition such that $F[x]$ is invertible. 1

(b) Prove that every Euclidean domain possesses unity. 2

(c) Show that $x^2 + 3x + 2$ has four zeros in Z_6. 2

(d) Let F be a field. Then prove that the ring of polynomial $F[x]$ is principal ideal domain (PID). 4

(e) Prove that a polynomial of degree n over a field has at most n zeros, counting multiplicity. 6

Or

Let F be a field and let $f(x), g(x) \in F[x]$ with $g(x) \neq 0$. Then prove that there exist unique polynomials $q(x)$ and $r(x)$ in $F[x]$ such that $f(x) = g(x)q(x) + r(x)$ and either $r(x) = 0$ or $\deg r(x) < \deg g(x)$.

2. *(a)* What is the inverse of $1+\sqrt{2}$ in $Z[\sqrt{2}]$? 1

(b) Define Euclidean domain. 1

(c) Test the irreducibility of the polynomial $x^5 + 9x^4 + 12x^2 + 6$ in Q. 2

(d) Prove that in a principal ideal domain, an element is irreducible if and only if it is a prime. 5

(e) Define unique factorization domain and prove that every field is unique factorization domain. 1+5=6

Or

Prove that $Z[\sqrt{3}] = \{a + b\sqrt{3} \mid a, b \in Z\}$ is a Euclidean domain. 6

3. *(a)* Write when two linear functionals are said to be equal on a vector space $V(F)$. 1

(b) Define invariant subspace. 1

(c) If S_1 and S_2 are two subsets of a vector space $V(F)$ such that $S_1 \subseteq S_2$, then show that $S_2^{\circ} \subseteq S_1^{\circ}$. 2

(d) Prove that the subspace spanned by two subspaces each of which is invariant under some linear operator T, is itself invariant under T. 3

(e) Let V be an n-dimensional vector space over the field F and let

$$\beta = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$$

be a basis for V. Then prove that there is a uniquely determined basis

$$\beta' = \{f_1, f_2, \cdots, f_n\}$$

for V' such that $f_i(\alpha_j) = \delta_{ij}$. 6

Or

Let V be finite dimensional vector space over the field F and let W be a subspace of V. Then prove that

$$\dim W + \dim W^{\circ} = \dim V$$

4. (a) Write about the eigenvalues and eigen-vectors of the identity matrix. 1

(b) If V is n-dimensional vector space, then what is the condition that the linear operator T is diagonalizable? 1

(c) Test the diagonalizability of the following matrix : 4

$$\begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

(d) Define minimal polynomial and show that the minimal polynomial of the real matrix

$$\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$$

is $(x-1)(x-2)$. 1+5=6

Or

If $f(x)$ be the characteristic polynomial of T, then prove that $f(T) = \hat{0}$. 6

5. (a) Write the only vector that is orthogonal to itself. 1

(b) Define orthogonal complement. 1

(c) If α, β are vectors in an inner product space V, then prove that

$$\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$$ 4

Or

If W_1 and W_2 are subspaces of a finite dimensional inner product space, then prove that

$$(W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}$$

(d) If $\beta = \{\alpha_1, \alpha_2, \cdots, \alpha_m\}$ is any finite orthonormal set in an inner product space V, and if β is any vector in V, then prove that

$$\sum_{i=1}^{m} |(\beta, \alpha_i)|^2 \leq \|\beta\|^2$$ 6

Or

In an inner product space, prove that

$$|(\alpha, \beta)| \leq \|\alpha\| \|\beta\|$$

6. (a) Write the two self-adjoint operators on any inner product space $V(F)$. 1

(b) Define normal operator. 1

(c) If T_1 and T_2 are normal operators on an inner product space with the property that either commutes with the adjoint of the other, then prove that $T_1 T_2$ is also normal operator. 2

(d) Let V be the direct sum of its subspaces W_1 and W_2. If E_1 is the projection on W_1 along W_2, and E_2 is the projection on W_2 along W_1, then prove that—

(i) $E_1 + E_2 = I$;

(ii) $E_1 E_2 = \hat{0}$, $E_2 E_1 = \hat{0}$. 4

(e) If T_1 and T_2 are self-adjoint linear operators on an inner product space V, then prove that (i) $T_1 + T_2$ is self-adjoint and (ii) if $T_1 \neq \hat{0}$ and a is a non-zero scalar, then aT_1 is self-adjoint iff a is real. 5

Or

Apply the Gram-Schmidt process to the vectors $(1, 0, 0)$, $(1, 1, 0)$, $(1, 1, 1)$ to obtain an orthonormal basis for $V_3(R)$ with the standard inner product.

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